Original article

The diet problem, a mathematical approach

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Abstract. *Background and aim:* Differential equations have always been used to modelize physical phenomena from other branches of science: physics, biology, chemistry, engineering, computer science etc. The aim of this paper is to find a simple mathematical model that can describe the variation of weight depending on time and calories intake. The idea is simple and is based on the so-called Malthus mathematical model, an ordinary differential equation associated to an initial condition, which studies the growth of a population with respect to a certain phenomenon or under the influence of external/internal factors. *Methods:* The most basic and intuitive Malthus model is formalized as follows: given P=P(t) the function that describes the size of a population, the ordinary differential equation P'(t)=rP expresses the fact that the rate of change of the size of the population (i.e. the derivative P'(t) with respect to the time t) depends directly on the size of the population itself multiplied by a factor r that represents the population growth rate, sometimes called Malthusian parameter. The equation needs to be associated to an initial condition, say $P_0 = P(0)$, which represents the size of the population at the time t=0. The solution of this problem can be calculated explicitly and this allows to precisely link the weight loss (or gain) according to calories intake, expected time, gender, kind of physical activity etc. *Results:* Our model considers age, gender and physical activity and allows us to discuss how to calculate a reasonable diet plan depending on different variables. Morever, it can give an idea, by studying the asymptotic behaviour of the solution, why the so-called miracle-diets can't work, why long diet plans usually fail and how to deal with severe obesity. *Conclusions:* The results obtained by means of this mathematical model shed new light on how to approach the creation of a reasonable diet plan. These results can be improved by introducing numerical simulations, which is the aim of a subsequent paper.

Key words: diet, mathematical model, math modelling, differential model, logistic equation

Historical background

The quest for a healthy way of living and hence the choice of a reasonable diet plan dates back to several centuries BC. The first known diet plan is claimed to be that of Charmis of Sparta, victor of the sprint race at the Olympic Games held in 668 BC. He is known to have trained following a diet of dried figs, founding the extra sugar intake helpful in sprints. A couple of centuries later, Dromeus of Stymphalus, who won the long race in 460 and 456 BC, based his fuel plan on meat. He was not the only one, as the wrestler Milo of Croton, six-time Olympic victor, was known for consuming as much as ten kgs of meat per day. Hence, it shouldn't be a surprise that the word *diet* derives from the Greek word *διαιτα* (*diaita*).

The first mathematician who followed a strict diet plan was Pythagoras. The word vegetarian got coined in the nineteenth century, and until then the word *Pythagorean diet* was used to describe a diet that excluded animal flesh. Pythagoras (?580–?500 BC) firmly believed in metempsychosis, the transmigration of souls. Hence, if souls did truly migrate from humans to animals, how could he touch meat? As a consequence, Pythagoras and his disciples followed a simple diet based on bread, vegetables and honey.

The nowadays famous *Hippocrates Diet*, formulated by Ann Wigmore (1909-1994), consists of uncooked living foods (fruits, vegetables, juices, sprouts, nuts and seeds) and fermented foods (miso and sauerkraut, raw honey). It actually takes inspiration directly by the book *On Regimen*, part of the *Hippocratic Corpus* (Latin: *Corpus Hippocraticum*), a collection of around 60 early Ancient Greek medical books usually associated with the physician Hippocrates (c. 460– c. 377 BC), though authorship is largely unknown. Hippocrates is often cited as one of the most outstanding figures in the history of medicine, and as the *Father of Medicine*. Still nowadays the *Hippocratic Oath* (or a slightly revised version of it) is an oath of ethics taken by physicians.

Hippocrates inspired also the famous *Taqwim al-Sihha*, known to be the first basic diet manual that included an index of 280 health-related items such as edibles, dishes, drinks, spices, lifestyles, and living enviroments, written around 1050 by Ibn Butlan (c. 1001–1066), a Christian physician in Baghdad.

More or less contemporary to this treatise is the *Regimen sanitatis salernitanum* (*Code of health of the school of Salernum*), a medieval didactic poem in hexameter verse. The true author is unknown, but the treatise is commonly attributed to John of Milan. It contains a description of practices and diet advices elaborated by the monks of the Salernum School in Italy.

The Italian humanist Bartolomeo Sacchi, known as Il Platina, wrote his *De honesta voluptate e valetudine*, the first bestselling diet book, in 1474. It quickly became widely popular throughout Europe, thanks to several reprints, and high society members carefully followed his advices on the relations between gastronomic pleasure (*voluptate*) and health (*valetudine*).

Shortly after Sacchi's book, Alvise (Luigi) Cornaro published in 1558 *Come vivere cento anni – Discorso della vita sobria* (How to Live One Hundred Years – Discourses on the Sober Life), a work that continues to be cited and sold even nowaydays, thanks to several translations and reprints. After conducting a self-indulgent lifestyle, the author, a Venetian aristocrat, decided to lead a healthy life, adopting a frugal diet. Some sources claim he lived up to one hundred years but, ironically, he died just eight years later the book was published, in 1566 at the age of 82, which was still a kind of record for that period. His book revives the Ancient Greek principles of moderation to prove that a healthy lifestyle relies mostly on a healthy diet.

More recently, Lulu Hunt Peters (1873-1930), an American doctor, published her best-selling book, *Diet and Health: With Key to the Calories* (1918), the first text to widely popularize the concept of counting calories as a method of weight loss. Her book was the main inspiration for other modern methods to lose weight, many of these being based on calories count. We will see later that pure and simple calories count isn't enough and is just a part of the equation.

Mathematical approach

Differential model

The aim is to find a simple model that can describe the variation of weight depending on time and calorie intake. The idea is simple and is based on the so-called *Malthus model*, which studies the growth of a population with respect to a certain phenomenon or under the influence of external/internal factors.

The most basic and intuitive Malthus model is formalized as follows: Given $P = P(t)$ the function that describes the size of a population, the ordinary differential equation

$$\frac{dP}{dt} = rP$$

expresses the fact that the rate of change of the size of the population (i.e. the derivative with respect to the time t) depends directly on the size of the population itself multiplied by a factor r that represents the population growth rate, sometimes called *Malthusian parameter*. The equation needs to be associated to an initial condition, say $P_0 = P(0)$, which represents the size of the population at the time $t = 0$.

The solution of this problem can be calculated explicitly and it gives $P(t) = P_0 e^{rt}$. This, according to Malthus, translates as "all life forms, including humans, have a propensity to exponential population growth when resources are sufficient but that actual growth is limited by available resources" (7). Several years later, in 1838,

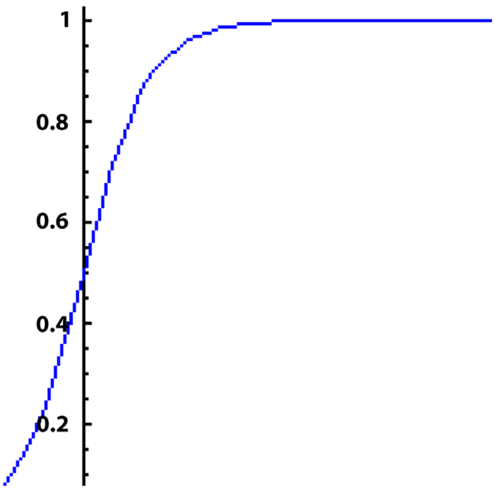

Figure 1. Insert figure caption here

a model of population growth bounded by resource limitations was developed by Belgian mathematician Pierre Francois Verhulst, after he had read Malthus' essay. Verhulst named the model a *logistic function*. More precisely, Verhulst developed the logistic function in a series of three papers between 1838 and 1847, based on his research on modeling population growth.

In this model the *logistic function* or *logistic curve* (see Figure 1) describes the growth of a given population P; at the beginning the growth is almost exponential, then slows down to reach a kind of asymptotic quasi-linear behaviour where growth is no longer attained.

This kind of functions is widely available in economy, biology, physics and other fields.

The associated equation is called *logistic equation* or *Verhulst model*; it is assumed that:

- the rate of growth is directly proportional to the existing population
- the rate of growth is directly proportional to the amount of available resources (e.g. food)

Assuming that P represents the size of the population and t is time, this model is formalized by the following first order ordinary differential equation:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

where r represents, as before, the rate of growth and K the asymptotic value of the population. Let us remark that when $K \to \infty$ the quantity $P/K \to 0$ and thus the Verhulst model becomes the Malthus model. The general solution for this equation is the logistic function. For a complete survey on the logit model we refer to (2).

We will apply the Malthus model to attack the diet problem, considering we need to study the rate of change (increase/decrease) of the weight of an individual, following a certain availability of resources (food/calorie). In order to do this we need to make certain assumptions and remarks.

1. with the word diet we refer to the set of all the nutrients (i.e. calories intake in various forms)
2. There are two types of calorie:
   - A small calorie (cal) is the amount of energy required to raise the temperature of 1 gram (g) of water by 1 Celsius.
   - A large calorie (kcal) is the amount of energy required to raise 1 kilogram (kg) of water by 1° C. It is also known as a kilocalorie. 1 kcal is equal to 1,000 cal. Normally we'll refer to calories when dealing with kcal.
3. a successful diet is ideally based on the relation between the energy intake ξ_a and energy consumption ξ_c, both measured in calories; if $\xi_a > \xi_c$ the diet plan causes a weight increase, if $\xi_a > \xi_c$ the diet plan causes weight loss. It should be remarked that this is a gross approximation, since it does not take into proper account the increasing energy expenditure with increasing weight or the reverse during weight loss. To be more precise, one should use a time-dependent relation, so that the rate of change of energy stores=rate of energy intake–rate of energy expenditure. Moreover, the relation between energy intake and energy expenditure assumes that the first law of thermodynamics holds true when applied to living beings. It has been noted, however, that different diet plans lead to different human body behaviours. Hence it is inappropriate to assume that the intake of calories is the only parameter to measure. In

other words, a calorie is certainly not a calorie as calories from carbohydrates are different, for human beings, from calories generated by proteins or fat. Hence, our model is a gross approximation of what happens during a diet plan and one should take into account where the calories come from, for example. For a review of these studies we refer to (4), (6) and (8) while for a basic differential diet model we refer to (9).

4. the estimated energy requirement (EER) represents the average dietary energy (caloric) intake that is predicted to maintain energy balance in a healthy adult of defined age, gender, weight, height, and level of physical activity, consistent with good health.

In order to calculate the EER we need to firstly determine the BMR (Basal Metabolic Rate). Metabolic rate represents the number of calories needed to fuel ventilation, blood circulation, and temperature regulation. Calories are also required to digest and absorb consumed food and fuel standard daily activities. Metabolic rate is an estimate of how many calories an individual would burn if he was to do nothing but rest for 24 hours. It represents the minimum amount of energy required to keep body functions.

BMR is synonymous with Basal Energy Expenditure or BEE. BMR measurements are typically taken in a darkened room upon waking after eight hours of sleep, 12 hours of fasting to ensure that the digestive system is inactive, and with the subject resting in a reclined position.

Sometimes the RMR (Resting Metabolic Rate) is used instead. RMR is synonymous with Resting Energy Expenditure or REE. RMR measurements are typically taken under less restricted conditions than BMR and do not require that the subject spends the night sleeping in the test facility prior to testing.

Both BMR and RMR are measured by gas analysis through either direct or indirect calorimetry, but a rough estimation of RMR or BMR can be acquired through the Mifflin-St.Jeor equation (see):

- Woman: $BMR = (9,99 \cdot M) + (6.25 \cdot H) - (4,92 \cdot T) - 161$
- Man: $BMR = (9,99 \cdot M) + (6,25 \cdot H) - (4,92 \cdot T) - 5$

where M is measured in kgs, H is measured in cms, and T represents the age of the individual.

In order to get the desired EER we just need to multiply the BMR by a factor that depends on the kind of physical activity of the individual. More precisely:

- $BMR \cdot 1.2$: low-intensity (sedentary) lifestyle
- $BMR \cdot 1.3$: medium-intensity (moderately active) lifestyle
- $BMR \cdot 1.4$: high-intensity (active) lifestyle

Note:

- Sedentary means a lifestyle that includes only the physical activity of independent living.
- Moderately Active means a lifestyle that includes physical activity equivalent to walking about 2.5 to 4.5 kms per day at 4.5 to 6.5 km/h, in addition to the activities of independent living.
- Active means a lifestyle that includes physical activity equivalent to walking more than 4.5 km/day at 4.5 to 6 km/h, in addition to the activities of independent living.

In this way we get the usual ideal weight chart (see table 1):

The weight of an individual depends on:

- daily calories intake (C)
- energy consumption

Calorie needs per kilogram highly vary by activity level, with people typically needing between 27-33 calories per kg if they're sedentary, 33-37 per kg if they're moderately active, and 37-40 per kg if they're very active. These ranges depends on gender, with women being at the lower end and men at the higher end. This means that, in order to maintain weight, people need somewhere between 26.4 and 39.6 calories per kilogram. If we wish to build a model that works averagely for everyone we can assume that an average calorie need to be $35cal$ per day. Hence, an individual that weighs W kg uses $35W$ cal per day.

Therefore, if

- $C = 35W$ weight remains constant
- $C < 35W$ weight decreases
- $C > 35W$ weight increases

Table 1: Weight chart

Men, T>25		Women, T>25	
Height (m)	**Ideal weight (kg)**	**Height (m)**	**Ideal weight (kg)**
1.55	51–69	1.42	42–49
1.58	52–60	1.45	43–50
1.60	54–62	1.47	44–51
1.63	55–63	1.50	45–53
1.65	56–65	1.52	46–54
1.68	58–67	1.55	48–55
1.70	60–69	1.57	49–57
1.72	62–71	1.60	50–59
1.75	64–73	1.63	52–61
1.77	65–75	1.65	54–63
1.80	66–77	1.68	55–64
1.85	69–79	1.70	57–67
1.87	71–82	1.73	59–69
1.90	73–84	1.75	61–70
1.93	74–86	1.78	63–72

Actually, considering the differences are not subtle, we prefer to build two different models, one for women and one for men, choosing 32 and 38, respectively, as Average Calorie Needs (ACN). For simplicity's sake one could consider 35 as an average value that approximately works well for both women and men, but that's too much inaccurate.

In order to build our two models, suppose that:

- the weight W at the time t is $W(t)kg$
- the variation of weight with respect to time is directly proportional to the difference between calories intake C and $ACN.W$ (daily energy consumption). Mathematically:

$$\frac{dW}{dt} \propto \left(C - ACN \cdot W\right).$$

This is exactly a kind of Verhulst model.

Hence we can formulate an ordinary differential equation, inserting a proportionality constant A (to be determined):

$$\frac{dW}{dt} = A\left(C - ACN \cdot W\right)$$

A is a constant to be measured in kg/cal and to calculate its value we need to exploit the usual conversion factor (7700 cal = 1kg of fat). Hence: $A = \frac{1}{7700} kg/cal$.

Our ordinary differential equation becomes:

$$\frac{dW}{dt} = \frac{1}{7700}\left(C - ACN \cdot W\right)$$

In order to solve this equation, we suppose C to be constant, and this means the calories intake doesn't vary with time. The solution is:

$$e^{0,0052t} W\left(t\right) = \frac{C}{ACN} e^{0,0045t} + k$$

with k constant to be determined depending on some initial condition. Let's suppose that at the time $t = 0$ (the beginning of the diet plan) the weight of the individual is $W(0) = W_0$. Hence at the time $t = 0$: $k = W\left(0\right) - \frac{C}{ACN}$.

The following relation expresses how weight varies depending on time:

$$W\left(t\right) = \frac{C}{ACN} + \left(W_0 - \frac{C}{ACN}\right) e^{-0,0045t}$$

Let us build our two models, one for women and one for men, inserting the real values of *ACN*.

Women model

$$\frac{dW}{dt} = \frac{1}{7700}(C - 32W)$$

We get:

$$e^{0,0041t}W(t) = \frac{C}{32}e^{0,0041t} + k;\, k constant$$

Calculate explicitly the value of k:

for $t = 0; W_0 = \frac{C}{32} + k; k = W_0 - \frac{C}{32}$

Hence the female diet model will be:

$$W(t) = \frac{C}{32} + \left(W_0 - \frac{C}{32}\right)e^{-0,0041t}$$

Men model

$$\frac{dW}{dt} = \frac{1}{7700}(C - 38W)$$

We get:

$$e^{0,0048t}W(t) = \frac{C}{38}e^{0,0049t} + k;\, k constant$$

Calculate explicitly the value of k:

for $t = 0; W_0 = \frac{C}{38} + k; k = W_0 - \frac{C}{38}$

Hence the male diet model will be:

$$W(t) = \frac{C}{38} + \left(W_0 - \frac{C}{38}\right)e^{-0,0049t}$$

Examples and applications

1. Woman

- 30 years
- 1,65m
- $W_0 = 85kg$
- $C = 2000$
- $W(t) = 63kg$

$$W(t) = \frac{2000}{32} + \left(85 - \frac{2000}{32}\right)e^{-0,0041t}$$

$$W(t) = 62,5 + 22,5e^{-0,0041t}$$

For how long time has to be followed the above diet plan in order to reach the desired target weight?

$$63 = 62,5 + 22,5e^{-0,0041t}$$

$$t \approx 928\ days$$

2. Man

- 33 years
- 1,70m
- $W_0 = 83kg$
- $C = 2400$
- $W(t) = 69kg$

$$W(t) = \frac{2400}{38} + \left(81 - \frac{2400}{38}\right)e^{-0,0049t}$$

$$W(t) = 63,16 + 19,84e^{-0,0049t}$$

For how long time has to be followed the above diet plan in order to reach the desired target weight?

$$69 = 63,16 + 19,84e^{-0,0041t}\ t \approx 627\ days$$

Why miracle diets do not work

Many miracle diets claim one can lose as much as 1 kg per day. Let us apply this claim to our female model: 22kg in 22 *days*

- $W(t) = 63\ kg$
- $W_0 = 85\ kg$
- $t = 22$

The equation becomes: $63 = \frac{C}{32} + \left(85 - \frac{C}{32}\right)e^{-0,0041 \cdot 22}$

Hence:

C ≈ −556 Clearly, the daily calorie intake can't be negative! Hence, the miracle diet can't work.

Severe obesity

An individual who is more than 200 kg over his healthy body weight (Body Mass Index, $BMI \geq 40$) has severe obesity. Severe obesity has the greatest risk of other health problems. Suppose our individual has an initial weight of 305.5 kg and plans to lose 1kg per day. Hence:

- $W_0 = 305,5kg$
- $W(t) = 283,5\,kg$
- 22 days (that is, a severe weight loss of 1 kg per day!)

The equation becomes:

$$283,5 = \frac{C}{32} + \left(305,5 - \frac{C}{32}\right)e^{-0,0041 \cdot 22}$$

$$C \approx 1582$$

Such a low daily calorie intake is hardly applicable to people who suffer from severe obesity. Instead, a more realistic approach could be the following:

- $W_0 = 305,5kg$
- $W(t) = 282,5\,kg$
- 60 days

The equation becomes:

$$282,5 = \frac{C}{32} + \left(305,5 - \frac{C}{32}\right)e^{-0,0041 \cdot 60}$$

C ≈ 6392, which is much more reasonable, considering severe obese individuals use to eat as much as 15,000/20,000 calories per day.

A healthy target for loss is usually to achieve a weight loss of 0.2-1kg of body weight each week over six months, leading to a decrease of 5 to 10% in body weight from baseline. For medical guidelines on BMI, obesity and severe obesity treatment we refer to (3).

Asymptotes in a diet plan

One could plan to follow a diet plan forever. Mathematically, this is equivalent to put $t \to \infty$. What happens? Let's apply this to our first example (woman diet plan). The equation was:

$$W(t) = 62,5 + 22,5e^{-0,0041t}$$

$$\lim_{t\to\infty} W(t) = \lim_{t\to\infty}\left(62.5 + 22.5e^{-0,0041t}\right) = 62.5$$

Hence $W(t) \to 62.5$ from above; in other words

$$\inf_{t \in \mathbb{R}^+} W(t) = 62.5$$

In other words, the woman's weight of our example 1) will asymptotically tend to 62.5 kg.

A mathematical reason why long diet plans usually fail

As we can gather from our models, the weight loss doesn't follow a linear decrease. People who lose 200g/day of weight the first month of their diet plan, expect to lose the same amount of weight the following months and...this is not the case! Actually, if we measure the average weight loss of our 1) and 2) examples, we get around 25 grams per day. The weight loss is noticeable in the first period and then becomes less effective. Hence, the reason why so many people tend to quit serious diet plans quite early is that, in the end, they believe that weight loss is a linear function, while it decreases as e^{-t}.

Further projects and conclusions

Quite surprisingly, mathematics and food have many things in common, for example a mathematical model on food and wine pairing can be found in (1). In this paper we have linked ordinary differential equations (and the Malthus/Verhulst model) to the diet problem. The aim of this paper was to build a model for a real world problem in such a way that it might be used as a powerful myth-debuking tool. In other words we have discovered how diet plans might work and why they can't work, especially when they promise "miraculous" results. The model adopts several gross approximations but it can be enriched and improved with the aid of dietologists and other experts in the field of human nutrition.

For example, a first step could be that of allowing the daily calories intake C to be variable and not constant. This will allow a more flexible application to different diet plans based on the concept that "a calorie is not just a calorie". Moreover, the model could take into proper account the fact that losing (or gaining) weight has an impact on the amount of consumed energy (moving a lighter object requires less energy).

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